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Linear or nonlinear hyperbolic wave problems with input sets (Part I)*

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SUMMARY

In this first part, collections of linear hyperbolic initial boundary value problems are treated which are defined via sets of coefficient functions in the differential equations. If the solutions are oscillatory, the nonlinear dependency of the solutions on coefficients becomes more and more ill-conditioned as time progresses, unless there is a sufficiently strong damping term in the differential equation. For the problem of dynamic buckling, the theory of the Neumann series yields a sufficient condition for the uniform boundedness of the oscillatory solutions which are induced by arbitrary continuous transient perturbations whose range is restricted to a suitable interval. In this part I, there is an introductory discussion of the Taylor-representation of sets of solutions in terms of constant coefficients. Via such a Taylor-representation, it is shown that solutions of the 'distortionless' telephone line are insensitive to sufficiently small variations of the constant coefficients in this hyperbolic differential equation.

1. Introduction

The widespread occurrence of wave phenomena in geology, mechanics, optics, physics, astronomy and a variety of engineering fields is well-known. A substantial number of these problems are modeled by hyperbolic systems, but probably the majority are not hyperbolic (e.g. water waves, dispersive waves). But, regardless of the equation type, the difficulties created by important physical and geometric nonlinearities are substantially multiplied by the fact that in real-world problems the input (physical) data are generally poorly known. All too often the model equations, hyperbolic or not, possess boundary and initial data, coefficients and perhaps forcing functions which cannot be defined as precisely specified values or functions. In such cases it may be possible to identify *input sets* which contain a particular input property in every possible case or, at least, in a substantial majority of the cases.

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Input sets are assumed to be well-defined in the sense that they possess well-defined boundaries in a finite-dimensional space or of the range of a set of functions. The input sets considered here consist of (a) every real vector in a compact subset of a Euclidean space, or (b) every continuous function on a certain domain with perhaps a fixed maximum Lipschitz constant or (c) a collection of special functions such as, e.g.

$$\sum_{K=1}^{N} \alpha_K \sin \beta_K x \quad \text{with every } \beta_K \text{ and/or } \alpha_K \text{ from given real intervals.} \tag{1.1}$$

It will be shown in Sections 2 and 3 that the admission of every (Lipschitz) continuous data or coefficient function, whose range is in certain intervals, may induce an instability such as resonance or parameter resonance, provided the solutions are oscillatory. Consequently, it is a difficult task to select a set of comparison data or coefficients which do not contain special functions yielding such an instability, unless the real-world problem being treated also is subject to these difficulties. Because of this reason, it may be desirable to restrict the admitted comparison functions to special collections such as in (1.1).

The main purpose of the (second part of the) paper is the development of methods for the construction of a sharp outer approximation of the range of a set of solutions. If the problems under consideration are inverse monotone, this task can be resolved by use of differential inequalities and related theories such as that of *M*-matrices, see [11], [13], and [9]. Since the class of inverse monotone operators is rather restricted as compared with the multitude of mathematical models of real-world problems, it was necessary to develop appropriate constructive methods. The methods are based in particular on (i) the theory of the *Neumann series* in the case of linear operator equations, (ii) *Taylor-expansions* in the case of constant parameters, and (iii) *boundary mapping* of the boundary of a data set onto the boundary of a set of solutions, if this mapping holds.

Even in the case of a collection of linear operator equations

$$A(z)u = f \quad \text{with every } z \text{ from an admitted set,} \tag{1.2}$$

the dependency of the solutions u on the coefficients z is nonlinear. In the majority of cases, transient waves are represented by oscillatory solutions, and the combination of this type of solution with the occurrence of sets of coefficients generally causes the nonlinear dependency of the solutions on the coefficients to become more and more ill-conditioned as time progresses, i.e., very small changes of z cause large changes of u. This nonlinear dependency of u on z (at fixed values of the independent variables) is perhaps a more serious difficulty in the development of efficient constructive methods than the nonlinear dependency of u on the data f if A is nonlinear.

Even in the case of an explicit representation of a set of solutions on both the independent variables and the coefficients, it is generally difficult to determine the range of the set of solutions or to detect special properties of the dependency of the solutions on coefficients. It is demonstrated in Section 6 that these difficulties can be resolved via a Taylor-representation of the set of solutions in terms of the coefficients if these are constant (parameters). If the solutions are *not* known *explicitly*, the dependencies on such parameters can still be treated via a

Taylor-representation, as will be shown in the second part. An appropriate estimate of the remainder term yields a quantitative error estimate for a truncation of the Taylor-expansion.

In the majority of real-world problems, the choice of the bounds of the input sets is as difficult as the choice of a fixed 'nominal' input. In such cases, it is desirable to execute a quantitative sensitivity analysis of the dependency of the solutions u on the coefficients or data. In this analysis, nested input sets may be assumed which contain the nominal coefficients or data. For each combination of fixed choices of sets of coefficients and data, an outer approximation of the set of solutions is constructed. Consequently, a generally nonlinear relationship between the ranges of the input sets and the output set of solutions is constructed which reveals quantitatively the sensitivity of the dependency of u on z and f. As an application, in Section 4, the sensitivity of the solution of a dynamic buckling problem with respect to transient periodic perturbations is analyzed and, in particular, sufficient conditions on the parameters are given to exclude parameter resonance.

A sensitivity analysis can also be carried out (see Part II) to study the dependency of the solution of a discrete analogy on the local discretization error. Such an estimate of the relationship between the local and the global discretization error is useful since (i) the customary criteria for stability and convergence do not yield a quantitative relationship between these errors and (ii) such criteria are almost entirely unknown for discrete analogies of nonlinear wave problems.

2. On ill-conditioned properties with respect to data

A complete sensitivity analysis of a mathematical model demands a quantitative comparison of a 'basic solution', $u(x,\gamma_0)$, pertaining to a given coefficient or data, $\gamma_0(x)$, with a set of 'neighboring solutions', $u(x,\gamma)$, referring to neighboring coefficients or data, $\gamma(x)$, from a set S_γ. Here, x stands for any number of independent variables. A set, S_γ, of 'comparison functions' γ has to be selected suitably such that $\gamma_0 \in S_\gamma$. The dependency of u on γ is ill-conditioned if small changes of γ yield large changes of u, which obviously causes difficulties in the application of constructive methods.

The quantitative sensitivity analysis would be considerably simplified if the values of the range of solutions, pertaining to S_γ, were bounded (globally or piecewise) by special solutions pertaining to the boundaries, γ_- and γ_+, of the range of admitted functions $\gamma \in S_\gamma$. Unfortunately, this desirable property holds true generally only for inverse-monotone operators and then generally only if γ stands for data. Only in that case, interval mathematics yields sharp bounds of the range of the set of solutions by relating bounds of this range to only γ_- and γ_+. If the problem is not inverse monotone, interval mathematics [8], [1] generally yields overestimates or underestimates of the range of the set of solutions, as will be shown.

The following hyperbolic ibvp (initial boundary value problem) is considered as an example:

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = g_0(t)\sin x \quad \text{for} \quad (x,t) \in \{(x,t)\mid 0<x<\pi, t>0\},$$

$$U(x,0) = u_0 \sin x, \qquad \frac{\partial U(x,0)}{\partial t} = u_1 \sin x \quad \text{for} \quad x \in [0,\pi], \quad u_0, u_1 \in \mathbb{R}, \tag{2.1}$$

$$U(0,t) = U(\pi,t) = 0 \quad \text{for} \quad t \in J := (0,\infty), \quad g_0 \in C(J).$$

A quantitative sensitivity analysis will be carried out now with respect to the forcing function $g_0 \sin x$. A separation of variables yields

$$U(x,t) =: u(t)\sin x \Rightarrow \begin{cases} u'' + u = g_0(t) & \text{for} \quad t \in J, \\ u(0) = u_0, u'(0) = u_1. \end{cases} \tag{2.2}$$

Here, $a := b$ or $b =: a$ means that a is defined to be b. For the purpose of the sensitivity analysis, (2.2) is embedded in the following collection of ivp (initial value problems) with a set S_g of comparison functions:

$$\begin{aligned} &u'' + u = g(t) \quad \text{for} \quad t \in J, \quad u(0) = u_0, \quad u'(0) = u_1, \\ &\text{every } g \in S_g := [g_-, g_+] \cap C(J), \text{ such that } g_0 \in S_g. \end{aligned} \tag{2.3}$$

The set of solutions, S_u, is a subset of the set $C^2(J)$.

i. *Construction of bounds for the range of the set of solutions*

For any fixed $g \in S_g$, the corresponding ivp in the collection (2.3) possesses the solution

$$u(t) = \int_0^t G(t,s) g(s) ds + \sigma(t)$$
$$\text{where} \begin{cases} G := \sin(t - s), \\ \sigma := u_0 \cos t + u_1 \sin t. \end{cases} \tag{2.4}$$

A function $u_+ \in C(J)$ is defined as follows:

$$u_+(t) := \int_0^t G(t,s)\, g_t(s)\, ds + \sigma(t) \text{ where}$$
$$Gg_t := \sup_{g \in S_g} (G(t,s)\, g(s)) = \begin{Bmatrix} g_+ G \text{ if } G(t,s) \geqslant 0 \\ g_- G \text{ if } G(t,s) < 0 \end{Bmatrix} \text{ locally for } (t,s) \in J \times J \text{ fixed.} \tag{2.5}$$

Because of

$$G(t,s) g(s) \leqslant \sup_{g \in S_g} (G(t,s) g(s)), \quad \text{every } (s,t) \in J \times J, \text{every } g \in S_g, \tag{2.6}$$

there holds

$$u(t) \leqslant u_+(t) \quad \text{for every } t \in J \text{ and every } g \in S_g, \tag{2.7}$$

with u_- correspondingly defined via inf instead of sup in (2.5).

ii. *The bounds u_+ and u_- are sharp*
For any fixed $\hat{t} \in J$, the definition of g_t implies that the discontinuous changes of g between g_+ and g_- take place precisely at those equidistant points $s(\hat{t})$ where $G(\hat{t}, s(\hat{t}))$ vanishes, i.e.,

$$G g_t \in C(J \times J), \tag{2.8}$$

see (2.5). For each $\hat{t} \in J$, there exist sequences of continuous functions $g_{\hat{t}}^{(\nu)}$ with $\nu \in \mathbb{N}$ such that

$$\lim_{\nu \to \infty} g_{\hat{t}}^{(\nu)}(s) = g_{\hat{t}}(s) \quad \text{for every } s \in J \backslash \{\hat{t}\} \text{ and every fixed } \hat{t} \in J. \tag{2.9}$$

For each such fixed $g_{\hat{t}}^{(\nu)} \in C(J)$ with $\nu \in \mathbb{N}$ fixed, the corresponding solution of (2.3) is

$$u_{\hat{t}}^{(\nu)} := \sigma + \int_0^{\hat{t}} G g_{\hat{t}}^{(\nu)} \mathrm{d}s \in C^2(J). \tag{2.10}$$

From (2.5) and (2.10),

$$\begin{aligned} &|u_+(t) - u_t^{(\nu)}| = \left| \int_0^t G(g_t - g_t^{(\nu)}) \mathrm{d}s \right| \Rightarrow \\ &|u_+(t) - u_t^{(\nu)}| \leqslant \sup_{(t,s) \in J \times J} |G(t,s)| \int_0^t |g_t(s) - g_t^{(\nu)}(s)| \, \mathrm{d}s. \end{aligned} \tag{2.11}$$

Even though the sequence $(g_t^{(\nu)})$ converges non-uniformly, the integral tends to zero as $\nu \to \infty$, i.e.,

$$\begin{aligned} &\lim_{\nu \to \infty} |u_+(t) - u_t^{(\nu)}| \leqslant \lim_{\nu \to \infty} \|g_t - g_t^{(\nu)}\|_1 \quad \text{for every } t \in J, \\ &\|\phi\|_1 := \int_a^b |\phi(s)| \, \mathrm{d}s \quad \text{for} \quad \phi : [a,b] \to \mathbb{R}. \end{aligned} \tag{2.12}$$

For every fixed $t \in J$, therefore, there is a solution $u_t^{(\nu)} \in S_u$ which for sufficiently large $\nu \in \mathbb{N}$ has an arbitrarily small distance from $u_+(t)$.

iii. *Comparison with a resonance-type problem in mechanics*
Without any loss of generality, it is assumed that g_- and g_+ are constant such that

$$\delta := -g_- = g_+ \in \mathbb{R}^+ \quad \text{for} \quad t \in J, \tag{2.13}$$

which is subsequently denoted as *Case I*. Then,

$$u_+ = \sigma + \delta S \quad \text{and} \quad u_- = \sigma - \delta S \quad \text{for } t \in J \text{ where}$$

$$S(t) := \int_0^t |G(t,s)|\,ds \quad \text{with} \quad g_t = \left\{\begin{matrix} \delta \text{ if } G(t,s) \geqslant 0 \\ -\delta \text{ if } G(t,s) < 0 \end{matrix}\right\} \text{locally} \tag{2.14}$$

$$\Rightarrow S(t) = 2m + \int_{m\pi}^t |\sin(t-s)|\,ds \quad \text{for} \quad t \in (m\pi, (m+1)\pi], \quad m \in \mathbb{N}.$$

For comparison, *Case II* is defined as follows:

$$u'' + u = \delta \sin t \quad \text{for} \quad t \in J, \quad u(0) = u_0, \quad u'(0) = u_1 \Rightarrow$$

$$u(t) = \sigma(t) + \delta\,[(-1)^{m+1} \frac{\pi}{2} m + \int_{m\pi}^t (\sin s)(\sin(t-s))\,ds] \tag{2.15}$$

$$\text{for} \quad t \in (m\pi, (m+1)\pi], \quad m \in \mathbb{N}.$$

Whereas the nonexistence of u as $t \to \infty$ in Case II is the classical resonance phenomenon, the nonexistence of u_+ as $t \to \infty$ in Case I may be denoted as *super-resonance*; the higher rate of increase of u_+ as compared with that of the maxima of u is due to the fact that u_+ is generated by selecting a suitable function $g_{\hat{t}}$ for every fixed $\hat{t} \in J$. As compared with the sharp bound u_+, an upper bound $\bar{u}_+$ of the range of the set of solutions u can be defined in the spirit of interval mathematics by use of an *independent* treatment of G and g in (2.4):

$$\bar{u}_+ := \sigma + \int_0^t \text{Max}\,\{G_- g_-, G_- g_+, G_+ g_-, G_+ g_+\}\,ds = \sigma + \delta \cdot t \text{ if } \delta = -g_- = g_+ \in \mathbb{R}^+$$

$$\text{where} \quad G_- = 0 \quad \text{and} \quad G_+ = -1. \tag{2.16}$$

Due to $2m < \pi m$, there holds $\bar{u}_+(t) > u_+(t)$ for $t > 0$; consequently, *this* application of interval mathematics to a non-inverse monotone problem yields an overestimate of the range of solutions.

iv. *Conclusions*

For any arbitrarily small $\delta \in \mathbb{R}^+$, the admission of every $g \in S_g := [-\delta, \delta] \cap C(J)$ causes the collection of solutions $U = u \cdot \sin x$ of (2.1), (2.2), (2.3) (a) to be unstable and unbounded if $t \in (0, \infty)$ and (b) to be ill-conditioned if $t \in [0, T]$, with a sufficiently large fixed $T(\delta) \in \mathbb{R}^+$.
Remark 1. Here, the span of the interval $[u_-, u_+]$ (t) has been maximized by use of the 'control functions' $g \in S_g$, and the g_t are the optimal policies realizing the maximum.
Remark 2. Due to damping (or a nonlinearity) in 'real' problems, there is usually no unbounded increase of u or u_+ as $t \to \infty$. If a linear damping law is additionally taken into account, the differential equation in (2.3) is replaced by

$$u'' + bu' + u = g \quad \text{for} \quad t \in J := (0, \infty), \quad b \in \mathbb{R}^+,$$
$$u(0) = u_0, \quad u'(0) = u_1, \quad \text{every } g \in S_g := [g_-, g_+] \cap C(J), \tag{2.17}$$

which yields

$$G(t, s) = \frac{2}{\lambda}\left(\exp(-b(t - s)/2)) (\sin(t - s) \frac{\lambda}{2}\right) \quad \text{for} \quad (t, s) \in J \times J, \quad \lambda = \sqrt{4 - b^2}. \tag{2.18}$$

The collection of ivp (2.3) is reconsidered for the case of $-g_- = g_+ \in \mathbb{R}^+$ only. The solution then is given by

$$u = g_+ + (u(0) - g_+) \cos t + u'(0) \sin t \quad \text{for} \quad t \in J. \tag{2.19}$$

For $g_- \in \mathbb{R}$, also a uniformly bounded solution is obtained. Consequently, an underestimate of the range of solutions is obtained by use of this naive application of interval mathematics which consists in a construction of u_- and u_+ via only the two elements g_- and g_+. Compare (2.16) for an appropriate employment of interval mathematics.

3. On ill-conditioned properties with respect to coefficients

The following hyperbolic ibvp is considered as an example:

$$\frac{\partial^2 U}{\partial t^2} - c_0^2(t) \frac{\partial^2 U}{\partial x^2} = 0 \quad \text{for} \quad (x, t) \in \{(x, t) \mid 0 < x < \pi, t > 0\}, \quad c_0^2 \in C(0, \infty),$$
$$U(x, 0) = 0, \quad \frac{\partial U(x, 0)}{\partial t} = \sin x \quad \text{for} \quad x \in [0, \pi], \tag{3.1}$$
$$U(0, t) = U(\pi, t) = 0 \qquad \text{for} \quad t \in J := (0, \infty).$$

A quantitative sensitivity analysis will be carried out now with respect to the coeffcient c_0^2. A separation of variables yields

$$U(x, t) =: u(t) \sin x \Rightarrow \begin{cases} u'' + c_0^2(t) u = 0 \quad \text{for} \quad t \in J := (0, \infty), \\ u(0) = 0, u'(0) = 1. \end{cases} \tag{3.2}$$

For the purpose of the sensitivity analysis, (3.2) is embedded in the following collection of ivp with a set of coefficients $c^2 \in S_c$:

$$u'' + c^2(t) u = 0 \quad \text{for} \quad t \in J, \quad u(0) = 0, \quad u'(0) = 1,$$
$$c^2 \in S_c := [c_-^2, c_+^2] \cap C(J) \quad \text{with } c_-, c_+ \in \mathbb{R} \text{ such that } c_-^2 < c_+^2 \text{ and } c_0^2 \in S_c. \tag{3.3}$$

i. *A special solution of the collection of ivp*

Just as in Section 2, a step function is chosen:

$$c_t^2 := \begin{cases} \hat{c}_{2\mu}^2 := c_-^2 & \text{for} \quad t \in (t_{2\mu}, t_{2\mu+1}], \\ \hat{c}_{2\mu+1}^2 := c_+^2 & \text{for} \quad t \in (t_{2\mu+1}, t_{2\mu+2}], \end{cases}$$

$$t_0 = 0, \quad c_-(t_{2\mu+1} - t_{2\mu}) = c_+(t_{2\mu+2} - t_{2\mu+1}) = \frac{\pi}{2}, \tag{3.4}$$

with $\mu + 1 \in \mathbb{N}$.

The function c_t^2 can be approximated with arbitrary accuracy in the L_1-norm such that the elements of the approximating sequences belong to the set S_c. A sequence of auxiliary ivp is defined:

$$u_\nu'' + \hat{c}_\nu^2 u_\nu = 0 \qquad \text{for} \quad t \in (t_\nu, t_{\nu+1}],$$

$$u_{\nu-1}^{(i)}(t_\nu) = u_\nu^{(i)}(t_\nu) \qquad \text{for} \quad \nu + 1 \in \mathbb{N} \text{ and } i = 0 \text{ or } 1, \tag{3.5}$$

$$u_0(0) = 0, u_0'(0) = 1.$$

Via an induction, it is shown that there holds

$$u(t) = \begin{cases} \dfrac{(-1)^\mu q^\mu}{c_-} \sin(\hat{c}_{2\mu}(t - t_{2\mu})) & \text{for} \quad t \in [t_{2\mu}, t_{2\mu+1}], \\ \dfrac{(-1)^\mu q^\mu}{c_-} \cos(\hat{c}_{2\mu+1}(t - t_{2\mu+1})) & \text{for} \quad t \in [t_{2\mu+1}, t_{2\mu+2}], \end{cases} \tag{3.6}$$

with $\mu + 1 \in \mathbb{N}$ and $q := \dfrac{c_+}{c_-}$,

Clearly,

$$\lim_{t \to \infty} |u(t)| = \infty \quad \text{for the choice of } c_t^2 \text{ in (3.4)}. \tag{3.7}$$

The nonexistence of u in (3.7) is due to the fact that the switch from c_- to c_+ is carried out precisely at the times t when $u'(t) = 0$ and, thus, the kinetic energy of the oscillation vanishes. Then, c_+ is employed only for such an interval of time that $u(t) = 0$ is reached with a correspondingly large slope, as an initial condition for the continuation of the solution with c_-. The nonexistence of u in (3.7) also occurs if the switch from c_- to c_+ is not carried out at every time t when $u'(t)$ vanishes but rather only at the times $t_{2\mu}$ and $t_{2\mu+1}$, defined by

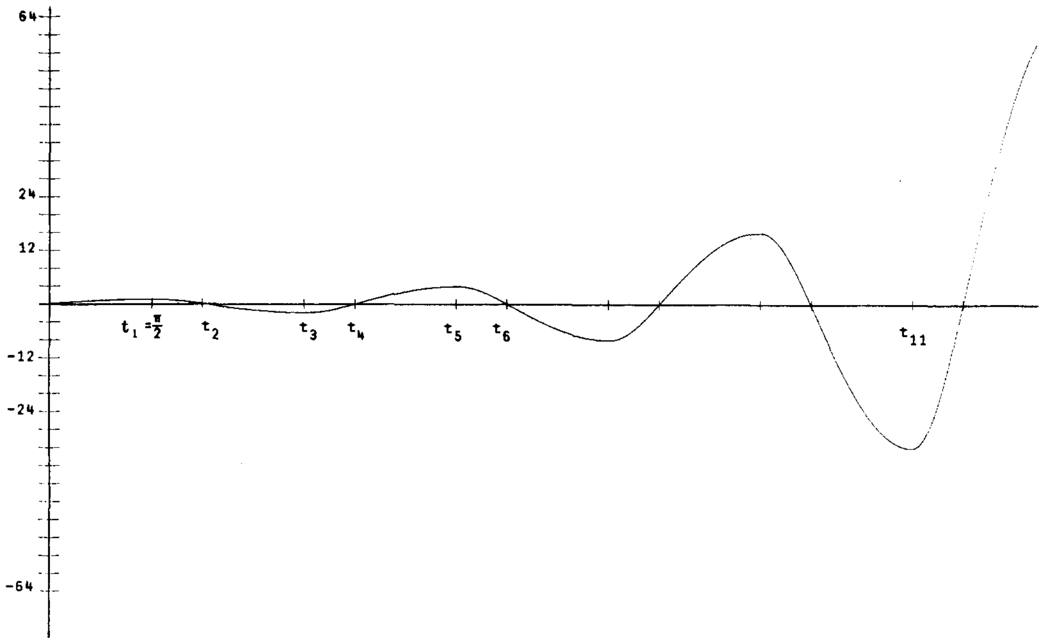

Figure 3.1a. Parameter resonance

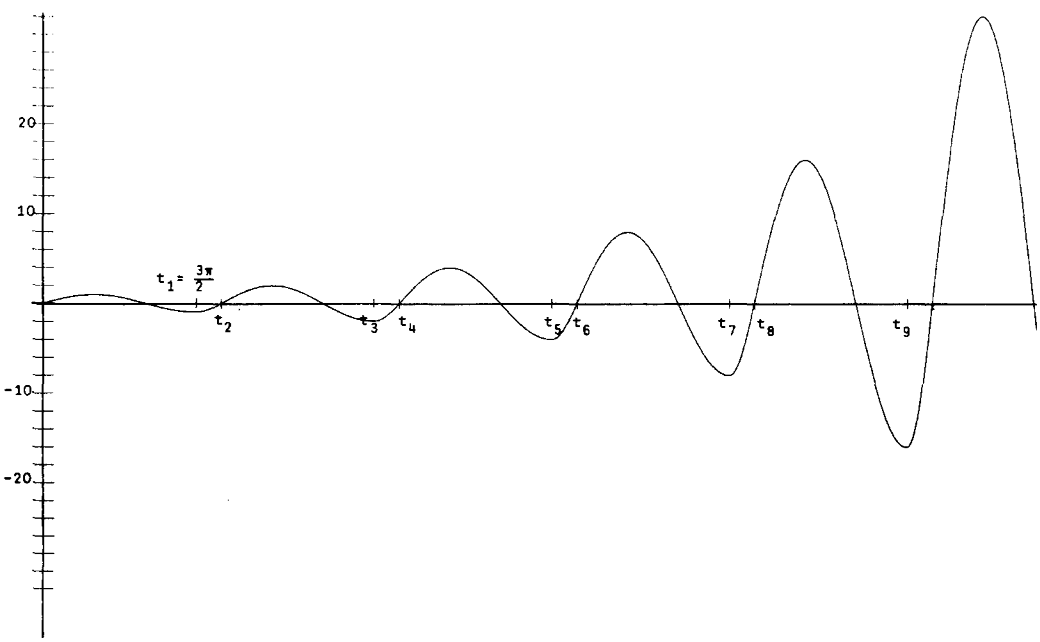

Figure 3.1b. Parameter resonance

$$c_-(t_{2\mu+1} - t_{2\mu}) = (2n+1)\frac{\pi}{2} \quad \text{with} \quad \mu+1, n+1 \in \mathbb{N},$$
$$c_+(t_{2\mu+2} - t_{2\mu+1}) = \frac{\pi}{2}, \tag{3.8}$$

see Fig. 3.1a for the initial stage of oscillation in the case of $n = 0$ and Fig. 3.1b for the case of $n = 1$, where $c_+ = 2$ and $c_- = 1$ have been chosen in both cases.

ii. *Comparison with the problem of parameter resonance in mechanics*

In many examples, the problem of parameter resonance (e.g., [3, p. 225-235] and [4]) can be reduced to the study of the Mathieu equation

$$u'' + \omega^2 (1 - \epsilon \cos \nu t)u = 0 \quad \text{for} \quad t \in J := (0, \infty)$$
$$\text{with} \quad u(0) = u_0, u'(0) = u_1; \omega, \epsilon, \nu, u_0, u_1 \in \mathbb{R}, \tag{3.9}$$

e.g., [6, p. 279-292]. The shaded areas in Fig. 3.2 represent domains of instability of the solution of (3.9). For any corresponding choice of ω, ν, ϵ, therefore, the limit of u does not exist as $t \to \infty$.
Remark 1. If there is an additional damping term, bu', in the differential equation in (3.9), then the width of the shaded areas in Fig. 3.2 tends to zero according to relations given in [3, p. 235]; in particular, there holds for the first shaded area, pertaining to $2\omega/\nu = 1$,

$$1 - \sqrt{\frac{\epsilon^2}{4} - 4\frac{b^2}{\nu^2}} < \left(\frac{2\omega}{\nu}\right)^2 < 1 + \sqrt{\frac{\epsilon^2}{4} - 4\frac{b^2}{\nu^2}}$$

$$\Rightarrow \text{stability if } \frac{\epsilon^2}{4} < 4\frac{b^2}{\nu^2}. \tag{3.10}$$

Remark 2. The results in Fig. 3.2 and (3.10) are due to truncations (with the terms of the third order) of series expansions of both $u(t, \epsilon, \nu)$ and $\omega^2(\epsilon, \nu)$; compare Section 4 for an independent analysis of the problem of parameter resonance.

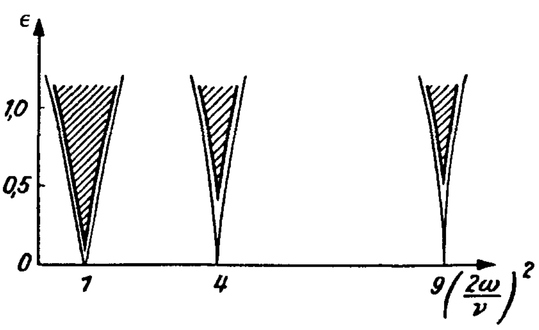

Figure 3.2. Stability diagram.

iii. *Comparison of parts (i) and (ii)*

The shaded regions of instability in Fig. 3.2 pertain to the following ratios $2\omega/\nu$:

$$\left(\frac{2\omega}{\nu}\right)^2 = (n+1)^2 \quad \text{with} \quad n+1 \in \mathbb{N} \Rightarrow$$
$$\frac{\lambda_c}{\lambda_u} = \frac{n+1}{2}, \qquad \text{where} \quad \lambda_c := \frac{2\pi}{\nu} \quad \text{and} \quad \lambda_u := \frac{2\pi}{\omega}. \tag{3.11}$$

Here, λ_c and λ_u are the wave lengths required for one full cycle of the functions $\omega^2(1 - \epsilon \cos \nu t)$ and u, respectively, see (3.9). Due to (3.8), the execution of the first complete cycle of the oscillatory function c_t^2 requires

$$\hat{\lambda}_c = c_- t_1 + c_+(t_2 - t_1) = \left(\frac{2n+2}{4}\right) 2\pi - \frac{\pi}{2} + \frac{\pi}{2} = \left(\frac{n+1}{2}\right) 2\pi, \tag{3.12}$$

where $\hat{\lambda}_c$ is the wave length of this first cycle. During this interval of time, $\hat{\lambda}_c$, the solution u executes $(n + 1)/2$ complete cycles, each of wave length $\hat{\lambda}_u$; i.e., the dependency on n of the solution u, due to switching between c_- and c_+, corresponds precisely to the numbers $(n + 1)^2$ characterizing the shaded areas in Fig. 3.2, see (3.11). A comparison of the special excitation modes c_t^2 due to (3.8) with Fig. 3.2 shows that there are non-denumerably many other coefficients $c^2 \in S_c$ yielding an unbounded response as $t \to \infty$.

iv. *General results for the ivp under discussion*

According to [2, p. 111-113], the solutions of

$$u'' + (1 + \phi(t))u = 0 \quad \text{for} \quad t \in J := (0, \infty) \text{ with}$$
$$\text{(i)} \ \lim_{t\to\infty} \phi(t) = 0 \quad \text{and} \quad \int_0^t |\phi(t)|\, dt < \infty \text{ or} \tag{3.13}$$
$$\text{(ii)} \ \phi \text{ tends monotonically to infinity as } t \to \infty,$$

are bounded as $t \to \infty$. The differential equations in (3.9) or in (3.3), (3.4) do not satisfy the conditions given in (3.13). The nonexistence of the limit of u as $t \to \infty$ in (3.9) or (3.3), (3.4), therefore, is due to the oscillatory character of the coefficients in (3.3) or (3.9) which persists as $t \to \infty$.

v. *Conclusions*

For any arbitrarily small $\delta := c_+^2 - c_-^2 \in \mathbb{R}^+$, the admission of every $c^2 \in S_c := [c_-^2, c_+^2] \cap C(J)$ causes the collection of solutions $U = u \sin x$ of (3.1), (3.2), (3.3) (a) to be unstable and un-

bounded if $t \in (0, \infty)$ and (b) to be ill-conditioned if $t \in [0, T]$ with a sufficiently large fixed $T(\delta) \in \mathbb{R}^+$. Here and in Section 2, it is concluded that a comparison of a given coefficient function with the functions in an embedding set of comparison coefficients involves the risk of thus admitting neighboring solutions which deviate arbitrarily from the given solution as $t \to \infty$.
Remark 3. Failure due to parameter resonance is also known to take place for classes of nonlinear ivp, [3] or [4]. By use of a separation of variables, this then immediately holds for suitable hyperbolic ibvp.

4. Sufficient conditions for the uniform boundedness of the response of a perturbed dynamic system

Transverse vibrations of a beam are considered which is subject to an axial compressive force, see Fig. 4.1. The flexural rigidity, EI, of the beam is assumed to be constant. The beam possesses the constant mass, m, per unit length and the damping constant, a, in a linear damping law, $a\ \partial y/\partial t$, of the transverse motion $y(x, t)$. Due to some external circumstances, the axial force is not stationary but rather may be subject to a time-dependent perturbation with small amplitude. This amplitude generally is not known quantitatively; however, it is possible to carry out a quantitative sensitivity analysis which admits every axial compression force in the following set:

$$P = P_0 + \hat{E}, \quad \text{every } \hat{E} \in S_{\hat{E}} := [-\hat{E}_0, \hat{E}_0] \cap C(J) \quad \text{with} \quad t \in J := (0, \infty), \tag{4.1}$$

$P_0 \in \mathbb{R}^+$ is fixed.

According to [4, p. 10], with the additional consideration of the damping term, this collection of problems possesses the mathematical model

$$EI\ \frac{\partial^4 y}{\partial x^4} + (P_0 + \hat{E})\ \frac{\partial^2 y}{\partial x^2} + a\ \frac{\partial y}{\partial t} + m\ \frac{\partial^2 y}{\partial t^2} = 0 \quad \text{for} \quad (x, t) \in G := \{(x, t) \mid 0 < x < L,$$

$$t \in J\}; \quad y(0, t) = y(L, t) = \frac{\partial^2 y(0, t)}{\partial x^2} = \frac{\partial^2 y(L, t)}{\partial x^2} = 0 \text{ for } t \in J; \tag{4.2}$$

$$y(x, 0) = \sin k\,\pi x/L, \quad \frac{\partial y(x, 0)}{\partial t} = 0 \quad \text{for} \quad x \in [0, L]$$

$EI, a, m \in \mathbb{R}^+$ are fixed, every $\hat{E} \in S_{\hat{E}}$.

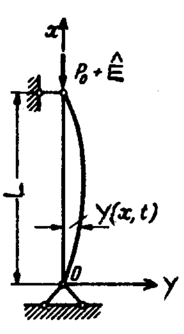

Figure 4.1. A dynamic buckling problem.

Here, a fixed initial position and a vanishing initial velocity have been assumed additionally.

The separation of variables,

$$y(x,t) =: u(t)\sin(k\pi x/L) \quad \text{for} \quad (x,t)\in G \tag{4.3}$$

satisfies (4.2) if there holds

$$mu'' + au' + (k^2\pi^2/L^2)(EI\,k^2\pi^2/L^2 - P_0 - \hat{E}(t))u = 0 \quad \text{for} \quad t\in J, \tag{4.4}$$

$$u(0) = 1, u'(0) = 0, \quad \text{every} \quad \hat{E}\in S_{\hat{E}}.$$

By use of simpler notations, (4.4) can be rewritten as follows:

$$\begin{aligned}&u'' + \alpha u' + \hat{\omega}^2 u = \omega^2 e(t)\,u \quad \text{for} \quad t\in J, \quad u(0) = 1, \quad u'(0) = 0,\\ &\alpha := a/m, \quad \omega^2 := P_0 k^2\pi^2/mL^2, \quad \hat{\omega} := (k^2\pi^2/L^2 m)(EI\,k^2\pi^2/L^2 - P_0),\\ &\text{every } e\in S_e := [-e_0, e_0]\cap C(J), \quad e_0 := \hat{E}_0 k^2\pi^2/mL^2 : J\to \mathbb{R}.\end{aligned} \tag{4.5}$$

The following three cases will be examined in detail:

$$\begin{aligned}&\text{(I)}\ \alpha = 0 \quad \text{and} \quad e(t) := e_\infty \exp(-\beta t) \quad \text{with } e_\infty\in\mathbb{R}, \beta\in\mathbb{R}^+ \text{ fixed},\\ &\text{(II)}\ \beta = 0 \quad \text{and} \quad \alpha, e_\infty\in\mathbb{R}^+ \text{ are fixed},\\ &\text{(III)}\ \text{suitable combinations of } \alpha, \beta, e_\infty\in\mathbb{R}^+ \text{ are admitted.}\end{aligned} \tag{4.6}$$

Remark. Case (I) satisfies the condition (i) in (3.13) for every $\beta, e_\infty \in \mathbb{R}^+$. If α is sufficiently large, case (II) is consistent with stability according to Fig. 3.2. In the following analysis real intervals $[-\bar{e}_\infty, \bar{e}_\infty]$, $[0, \bar{\alpha}]$, and $[0, \bar{\beta}]$ will be identified such that parameter resonance can be excluded for every combination of e_∞ α, and β from these intervals. Correspondingly, the dynamic buckling problem then is insensitive with respect to these admitted perturbations.

By use of the left hand side in (4.5), Green's function is

$$G(t,s) := \frac{2}{\lambda}\exp(-\alpha(t-s)/2)\sin\frac{\lambda}{2}(t-s) \quad \text{for} \quad (t,s)\in J\times J, \tag{4.7}$$

$$\lambda := \sqrt{4\hat{\omega}^2 - \alpha^2}.$$

The case of oscillatory (i.e. non-monotone) solutions holds if the following condition is satisfied:

$$4\hat{\omega}^2 > \alpha^2. \tag{4.8}$$

The collection of ivp (4.5) can be represented equivalently via the collection of Volterra integral equations

$$u(t) = \sigma(t) + \int_0^t G(t,s)\,\omega^2 e(s)u(s)\,ds \quad \text{for} \quad t\in J, \tag{4.9}$$

$$\sigma := \exp(-\alpha t/2)\cos\lambda t/2, \quad \text{every } e\in S_e,$$

or the collection of operator equations

$$
\begin{aligned}
&(I - F_e)u = \sigma \quad \text{where} \quad Iu := u, \quad \text{every } e \in S_e, \\
&F_e[u](t) := \omega^2 \int_0^t G(t,s)e(s)\,u(s)\mathrm{d}s \quad \text{for} \quad t \in J.
\end{aligned}
\tag{4.10}
$$

There holds

$$
|F_e[u](t)| \leqslant \frac{2\omega^2}{\lambda} \|u\|_\infty \int_0^t e^{-\alpha(t-s)/2} |e(s)|\,\mathrm{d}s \leqslant \|F_e\|_\infty \|u\|_\infty,
$$

$$
\|u\|_\infty := \sup_{t\in J} |u(t)| \Rightarrow \|F_e\|_\infty = \frac{2\omega^2}{\lambda} \sup_{t\in J} \int_0^t e^{-\alpha(t-s)/2} |e(s)|\,\mathrm{d}s = \tag{4.11}
$$

$$
\frac{2\omega^2}{\lambda} e_\infty \sup_{t\in J} \frac{e^{-\beta t} - e^{-\alpha t/2}}{\frac{1}{2}\alpha - \beta},
$$

due to the customary definition of the norm of an operator, i.e.,

$$
\begin{aligned}
&\textit{Case (I)}: \ \|F_e\|_\infty < 1 \text{ if } \alpha = 0 \quad \text{and} \quad \frac{2\omega^2 e_\infty}{2\hat{\omega}\beta} < 1; \\
&\textit{Case (II)}: \ \|F_e\|_\infty < 1 \text{ if } \beta = 0 \quad \text{and} \quad \frac{4\omega^2 e_\infty}{\left(\sqrt{4\hat{\omega}^2 - \alpha^2}\right)\alpha} < 1.
\end{aligned}
\tag{4.12}
$$

The condition $\|F_e\|_\infty < 1$ can be satisfied if $e_\infty \in \mathbb{R}^+$ is sufficiently small, irrespective of whether the damping constant $a \in \mathbb{R}^+$ is zero or not. Then, (4.10) can be solved via the Neumann series e.g. [10], for every admitted $e \in S_e$:

$$
\left.
\begin{aligned}
&u = (I + F_e + F_e^2 + \ldots)\sigma \quad \text{for} \quad t \in J \\
&\text{if } \|F_e\|_\infty < 1
\end{aligned}
\right\} \text{ for every } e \in S_e, \text{ every } t \in J. \tag{4.13}
$$

Since $q := \|F_e\|_\infty < 1$, the truncation of the series with the first term, σ, possesses the well-known error estimate $q\,\|\sigma\|_\infty\,(1-q)^{-1}$ which implies that the sum of this series is uniformly bounded for $t > 0$.

The set of functions S_e contains the special trigonometric functions

$$
\begin{aligned}
&\hat{E} = P_1 \cos \nu t \text{ with } P_1 \in \mathbb{R}^+ \text{ fixed and every } \nu \in \mathbb{N}, \\
&e = (k^2\pi^2/L^2 m)\,P_1 \cos \nu t,
\end{aligned}
\tag{4.14}
$$

which appear in the special realization of the mathematical model (4.2) governing the theory of parameter resonance.

This kinetic instability (see Section 3) can be excluded for every combination of the real parameters α, β, and e_∞ satisfying $\|F_e\|_\infty < 1$. This result admits every continuous perturbation $e \in S_e$ such that $|e| \leqslant \bar{e}_\infty \exp(-\beta t)$ for $t \in J$. In the literature ([3], [4]), the problem consisting of (4.2) and (4.14) is customarily solved approximately by use of asymptotic expansions, without any error estimate. In [3] and [4], quantitative results on domains of stability or instability are given only for ν not exceeding ten. A comparison with these results in the available literature reveals that the conclusions on uniform boundedness, as following from (4.12), hold generally for every $\nu \in \mathbb{N}$ and without any unaccounted procedural errors involved.
Remarks: 1). The choice of a very small positive constant β may be of interest in practical applications which usually are concerned with only a bounded interval of time t. 2) Correspondingly, the admission of Fourier expansions (if uniformly convergent) for $y(x, 0)$ or $\partial y(x, 0)/\partial t$ is tractable; 3) Mathematical models with periodically modulated coefficients are employed in electrical engineering, either ivp or hyperbolic ibvp (for the transmission of waves), e.g. [12, p. 466].

5. On Taylor-approximations of sets of solutions depending on constant coefficients

5.1 *Introduction to the problem*

Any mathematical model is considered which possesses oscillatory or wave-type solutions:

$$\begin{aligned} &A(z^{(0)})\, u = f \quad \text{for} \quad (x,t) \in D := D_0 \times J \subset \mathbb{R}^{n+1}, \quad J := (0, T] \subset \mathbb{R}, \\ &u, f\colon D \to \mathbb{R}^m, \quad u \in U_u, \quad f \in U_f, \quad A: U_u \times I_z \to U_f, \\ &z^{(0)} \in I_z \subset \mathbb{R}^p \text{ with } z^{(0)} \text{ fixed.} \end{aligned} \tag{5.1}$$

Here, t is a time-like independent variable and x represents one or several spatial independent variables; U_u and U_f are function spaces with suitable smoothness properties.

In many real-world problems, a mathematical model (5.1) is to be examined not only for the fixed parameter $z^{(0)} \in I_z$ but rather for every $z \in I_z$; hence, a collection of problems

$$A(z)u = f \quad \text{for} \quad (x, t) \in D, \quad \text{every } z = (z_1, \ldots, z_p)^T \in I_z \tag{5.2}$$

will be discussed subsequently. A particularly important motivation is the problem of a (quantitative) sensitivity analysis of the solution $u^{(0)}(x, t) := u(x, t; z^{(0)})$ of (5.1) with respect to (small) changes $z \in I_z$ of parameters such that $z \neq z^{(0)}$. If (5.1) represents the hyperbolic ibvp of the 'telephone line' (compare Section 6), the distortions of electro-magnetic signals are of interest which are due to small deviations of the real 'fabricated' parameters z_k from the intended 'design values' $z_k{}^{(0)}$.

In real-world problems, the consequences of an uncertainty of $z^{(0)}$ usually cannot be investigated completely via the prescription of an interval $I_z \ni z^{(0)}$ since there is usually not enough pertinent information available. If this problem is treated in the context of a stochastic theory, probability values are chosen to measure the probability of the events that the z_k deviate from $z_k^{(0)}$ by a certain magnitude. This problem can be treated alternatively by examining a nested sequence of intervals $I_z^{(r)}$ with $r = 1(1)q$ and $q \in \mathbb{N}$ fixed such that there holds $z^{(0)} \in I_z^{(r)}$ for $r = 1(1)q$ and each $I_z^{(r)}$ is characterized by a fixed probability parameter. This yields a nested sequence of sets of solutions.

For practical purposes, it is usually sufficient to have quantitative information on the generally nonlinear function $u(x, t; z)$, for fixed choices of $(x, t) \in D$. This quantitative sensitivity analysis can be carried out in a conservative fashion by the construction of outer approximations for the envelopes of the sets of solutions pertaining to several *choices* of nested intervals $I_z^{(r)}$ as introduced above.

Since $z \in \mathbb{R}^p$, it is possible for any fixed $(x, t) \in D$ to represent $u(x, t; z)$ via a Taylor-polynomial with remainder term provided that suitable smoothness conditions are satisfied. This approach yields a local approximation of $u(x, t; z)$ as a function of z in a neighborhood of $z^{(0)}$. If the remainder term is ignored, this is a customary perturbation analysis.

5.2. *On Taylor-representations of the solutions as functions of the parameters*

It is assumed that the solution $u(t, u; z)$ of (5.1) possesses the following properties:

(i) u exists uniquely for every $(x, t; z) \in D \times I$;
(ii) at every fixed $(x, t) \in D$, there holds $u \in C^{(q+1)}(I_z)$ with a fixed $q + 1 \in \mathbb{N}$. (5.3)

At any fixed $(x, t) \in D$, then, $u(x, t; z)$ can be represented via the following Taylor-approximation with remainder term R_{q+1}:

$$u(x, t, z) = u(x, t, z^{(0)}) + [(z - z^{(0)}) \cdot \nabla]\, u(x, t; z^{(0)}) + \dots +$$
$$+ [(z - z^{(0)}) \cdot \nabla]^q u(x, t; z^{(0)}) + R_{q+1}(x, t; z, z^{(0)})),$$

where

$$(z - z^{(0)}) \cdot \nabla := \sum_{j=1}^{p} (z_j - z_j^{(0)}) \frac{\partial}{\partial z_j},$$

(5.4)

$$R_{q+1} := \frac{1}{(q+1)!} \begin{pmatrix} [(z - z^{(0)}) \cdot \nabla]^{q+1} u_1(x, t; z^{(0)} + J_1(z - z^{(0)})) \\ \vdots \qquad\qquad \vdots \\ [(z - z^{(0)}) \cdot \nabla]^{q+1} u_m(x, t; z^{(0)} + J_m(z - z^{(0)})) \end{pmatrix},$$

with $J_j(x, t) \in (0, 1)$ for $j = 1(1)m$.

The validity of this local approximation at $z^{(0)}$ in terms of the $z_j - z_j^{(0)}$ depends (i) on the proximity of the solutions $u(x, t; z)$ for every $z \in I_z$ with $(x, t) \in D$ fixed and, therefore, on the magnitude of R_{q+1} and (ii) on the possibility to come up with a meaningful quantitative estimate of R_{q+1}.

The construction of (5.4) generally cannot be executed if the operator A in (5.1) contains partial differential equations. This is not so if (5.1) employs only ordinary differential equations or if the dependency of the solution on parameters is known explicitly. As an example, (5.1) is assumed to be a collection of ivp:

$$u'_1 = f_i(t,u,z) \quad \text{for} \quad t \in J, \quad i = 1(1)n, \quad z \in I_z \in \mathbb{R}^p, \quad u = (u_1, \ldots, u_n)^T: \\ J \times I_z \to \mathbb{R}^m, \quad f = (f, \ldots, f_n)^T \in C^{(1,\hat{q},\hat{q})}(J \times \mathbb{R}^m \times I_z) \quad \text{where} \quad \hat{q} = 1 + q. \tag{5.5}$$

According to [7, p. 35], then there hold for sufficiently small T in $J := (0, T]$.

$$\begin{aligned} &\text{(i) a unique solution } u(t,\hat{z}) \text{ with } u \in C^1(J) \text{ exists for every fixed } \hat{z} \in I_z \text{ and} \\ &\text{(ii) the solution } u(\hat{t}; z) \text{ satisfies } u \in C^{1+q}(I_z) \text{ for every fixed } \hat{t} \in J. \end{aligned} \tag{5.6}$$

The following example shows that the local approximation of solutions being discussed is only of restricted validity in the case of oscillatory solutions.

The collection of ivp

$$\begin{aligned} &u'' + z^2 u = 0 \quad \text{for} \quad t \in J := (0, \infty), \quad u(0) = 0, \quad u'(0) = 1, \\ &\text{every } z \in I_z := [1 - \delta, 1 + \delta], \delta \in \mathbb{R}^+ \text{ is arbitrarily small,} \end{aligned} \tag{5.7}$$

possesses the solutions $u = (\sin zt)/z$ and, therefore, the following Taylor-representation with respect to $z^{(0)} = 1$:

$$u(t;z) = \sin t + (z - 1)(t \cos t - \sin t) + \frac{(z-1)^2}{2}(-t^2 \sin t - 2t \cos t + 2 \sin t) \\ + \ldots \quad \text{for} \quad t \in J, \quad z \in I_z. \tag{5.8}$$

The ν-th term of this expansion, with $\nu + 1 \in \mathbb{N}$, contains a factor t^ν. If truncated with the ν-*th* term, the corresponding polynomial is a meaningful local approximation for $t \in C_0 := [0, T_0] \subset [0, \infty)$ with $T_0(\delta)$ sufficiently small. For $t \in C_m := (T_0, T_m] \subset [0, \infty)$, interpolations with respect to $z \in I_z$ may be used to construct a meaningful approximation of the solutions u. For $t \in C_\infty := [0, \infty) \backslash (C_0 \, U \, C_m)$, local approximations or interpolations fail entirely, and solutions $u(t; z)$ have to be studied separately for each individual $z \in I_z$.

The local approximations to be employed subsequently will be used only in the initial time-domain, C_0, whose extent depends on the span of the admitted parameter interval I_z. This extent can be increased if I_z is suitably partitioned and local approximations of the solutions are constructed separately for each subinterval of I_z.

6. Sensitivity of the linear telephone line

6.1 *Introduction*

In the discussion and analysis of telephonic transmission (Guillemin [5]) in a pair of linear conductors, the telephone equation

$$e_{\hat{x}\hat{x}} = LCe_{\hat{t}\hat{t}} + (LG + RC)e_{\hat{t}} + RGe, \tag{6.1}$$

stated for the voltage $e(x, t)$, is important. Here, $C, G, L,$ and R are capacitance, conductance, inductance, and resistance (all per unit length), respectively. By means of the scaling transformation $\hat{t} = at, \hat{x} = bx$, with $a = (RG)^{-1/2}$ and $b = (LG + RC)/RG$, equation (6.1) becomes

$$e_{xx} = \frac{\mu}{(1+\mu)^2} e_{tt} + e_t + e, \tag{6.2}$$

where $\mu := RC/LG$ is the third basic parameter and the one we shall be concerned with in the following work.

If $\mu = \mu_0 = 1$ this is the well-known *distortionless* line. A sensitivity analysis will be carried out in a neighborhood of $\mu = 1$ with a and b fixed (note that a and b only occur in the *dimensioned equation*) by means of a Taylor-series expansion in the parameter μ with remainder term,

$$\begin{aligned} e(x,t;\mu) &= e(x,t;1) + (\mu - 1)\frac{\partial e}{\partial \mu}(x,t;1) + \\ &+ \frac{1}{2!}(\mu - 1)^2 \frac{\partial^2 e}{\partial \mu^2}(x,t;1+\theta(\mu-1)), \quad \theta \in (0,1). \end{aligned} \tag{6.3}$$

Before starting the analysis we set

$$e(x,t;\mu) =: \exp(-(1+\mu)^2 t/2\mu)\, y(x,t;\mu), \tag{6.4}$$

whereupon the equation for y becomes

$$y_{xx} = \frac{\mu}{(1+\mu)^2} y_{tt} - \frac{(\mu-1)^2}{4\mu} y. \tag{6.5}$$

For the distortionless line the equation for y becomes $4y_{xx} = y_{tt}$. Then the general solution for e is

$$e(x,t;1) = \exp(-2t)\{\hat{f}(x+2t) + \hat{g}(x-2t)\},$$

where $\hat{f}$ and $\hat{g}$ are arbitrary C^2 functions.

6.2 *Sensitivity analysis of an exact solution for semi-infinite lines*

The auxiliary conditions to be used for (6.2) are

$$e(0,t;\mu) = f(t), \quad e(x,0;\mu) = e_t(x,0;\mu) = 0, \\ e(x \to \infty, t;\mu) \text{ is finite.} \tag{6.6}$$

They become the conditions

$$y(0,t;\mu) = \exp\left((1+\mu)^2 t/2\mu\right) f(t) =: g(t,\mu), \\ y(x,0;\mu) = y_t(x,0;\mu) = 0, \\ y(x \to \infty, t, \mu) \text{ is finite,} \tag{6.7}$$

for equation (6.5).

Designating $E(x, s; \mu)$, $Y(x, s; \mu)$ and $F(s)$ as the Laplace transforms of e, y and f, respectively, one easily finds that

$$E(x,s;\mu) = F(s)\exp\left[-\lambda\left(s + \frac{(1+\mu)^2}{2\mu}\right)x\right], \tag{6.8}$$

where the function λ^2 is given by

$$\lambda^2(r) := \frac{\mu}{(1+\mu)^2}\, r^2 - \frac{(\mu-1)^2}{4\mu}\,. \tag{6.9}$$

Upon setting $r := s + (1+\mu)^2/(2\mu)$ it is easily verified that

$$\lambda^2\left(s + \frac{(1+\mu)^2}{2\mu}\right) = \frac{\mu}{(1+\mu)^2}\, s^2 + s + 1 =: \omega^2(s;\mu). \tag{6.10}$$

Clearly $\omega^2 > 0$ for all real $s > 0$. In this notation (6.8) becomes $E(x,s;\mu) = F(s)\exp\left[-\omega(s;\mu)x\right]$. From this it is easily shown that $\partial\omega/\partial\mu\,|_{\mu=1} = 0$, whereupon there will not be any first order term (in $(\mu - 1)$) in (6.3) after inversion.

Because of the possibility for obtaining an exact solution here the second order term will be computed exactly. Subsequently, we will discuss an estimation for the second order terms, thought of as a *remainder* term, in the series.

A second computation shows that there is a second-order effect, whereupon

$$E(x, s; \mu) = F(s)\exp\left[-(s+2)x/2\right]\left\{1 + \frac{(\mu-1)^2}{8}\,\frac{xs^2}{s+2}\right\} \tag{6.11}$$

to second order in powers of $(\mu - 1)$.

To illustrate, suppose $f(t) = \sin t$. Then $F(s) = (s^2 + 1)^{-1}$ and

$$E = \exp\left[-(s+2)x/2\right]\left\{\frac{1}{s^2+1} + \left[\frac{(\mu-1)^2 x}{8}\right]\frac{s^2}{(s+2)(s^2+1)}\right\} + \cdots$$

which is to be inverted. From standard tables the inverse is easily found to be

$$e(x,t;\mu) = \exp(-x)\,u(t-x/2)\{\sin(t-x/2)\} \tag{6.12}$$

$$+\ \frac{(\mu-1)^2}{40}\,x\left[\exp(-2(t-x/2)) + \cos(t-x/2) - 2\sin(t-x/2)\right] + \cdots$$

accurate to terms of order two in $(\mu - 1)$. Here $u(t - x/2)$ is Heaviside's step function

$$u(t-x/2) := \begin{cases} 0, & t < x/2, \\ 1, & t > x/2. \end{cases}$$

Remark 1. The finite line with auxiliary conditions $e(0, t; \mu) = f(t)$, $e(1, t; \mu) = 0$, $e(x, 0; \mu) = e_t(x, 0; \mu) = 0$ can be treated in a similar fashion. In this case it is found that

$$E(x,s;\mu) = F(s)\sinh\{\omega(s;\mu)(1-x)\}/\sinh\omega(s;\mu) \qquad \text{where } \omega(s;\mu) \text{ was defined in (6.10).}$$

Remark 2. If a sensitivity analysis is also desired with respect to the parameters a and b of Section 6.1, additional terms in $(a - a_0)$ and $(b - b_0)$, where a_0 and b_0 are reference values, would arise. The computation is similar to that described above.

Remark 3. When the remainder form is used for E (see (6.3)) in the form

$$E(x,s;\mu) = E(x,s;1) + \frac{(\mu-1)^2}{2}\,\frac{\partial^2 E(x,s;1+\theta(\mu-1))}{\partial\mu^2}, \tag{6.13}$$

$\theta \in (0,1)$, the quantity θ is not known. Therefore, an interval evaluation of the second derivative in (6.13) is proposed for every $\mu \in [1 - \delta, 1 + \delta]$ with δ fixed, $0 < \delta \ll 1$. Since

$$\frac{\partial^2 E}{\partial\mu^2} = -xF(s)\left\{\exp(-\omega x)\,\frac{\partial^2\omega}{\partial\mu^2} - x\exp(-\omega x)\left(\frac{\partial\omega}{\partial\mu}\right)^2\right\}$$

$$=:\ -x\,F(s)\,\Omega(x,s;\mu),$$

it is sufficient to compute an interval for $\Omega(x, s; \mu) \in [\Omega_-, \Omega_+]$, where Ω_-, Ω_+ depend upon x and s only.

Remark 4. Following (6.10), it was shown via a differentiation of (6.8) that $\partial E/\partial\mu = 0$ for $\mu = 1$. If an explicit solution for E were not known, a linear hyperbolic problem for $z := \partial E(x,s;\mu)/\partial\mu$

could be derived by use of differentiations with respect to μ of the individual equations in (6.5) and (6.7). This problem for z possesses only the trivial solution in the case of $\mu = 1$, if $\lim y = 0$ as $x \to \infty$ for every $t = 0$. This vanishing boundary value of y is obtained automatically in the preceding construction of the explicit solution via the Laplace transform Y of the solution of (6.5), (6.7).

Further let $\omega \in [\omega_-, \omega_+]$, $\partial\omega/\partial\mu \in [\omega_-', \omega_+']$ and $\partial^2\omega/\partial\mu^2 \in [\omega_-'', \omega_+'']$, where $\omega_-, \omega_+, \omega_-', \ldots, \omega_+''$ only depend upon s.

The interval evaluation will be carried out for the following equivalent functional representation of Ω and every $\mu \in [1-\delta, 1+\delta]$:

$$\Omega = \frac{\exp(-\omega x)}{2\omega}\left\{-2\left(\frac{\partial\omega}{\partial\mu}\right)^2 + s^2\left(\frac{2\mu-4}{(1+\mu)^4}\right)\right\} - x\exp(-\omega x)\left(\frac{\partial\omega}{\partial\mu}\right)^2. \tag{6.14}$$

Equation (6.14) follows from the calculations of $\partial\omega/\partial\mu$ and $\partial^2\omega/\partial\mu^2$, which become

$$\partial\omega/\partial\mu = s^2(1-\mu)/\omega(1+\mu)^3$$

and

$$\partial^2\omega/\partial\mu^2 = \frac{1}{2\omega}\left\{-2\left(\frac{1-\mu}{(1+\mu)^3}\right)^2\frac{s^4}{\omega^2} + s^2\left(\frac{2\mu-4}{(1+\mu)^4}\right)\right\}$$
$$= \frac{1}{2\omega}\left\{-2\left(\frac{\partial\omega}{\partial\mu}\right)^2 + s^2\left(\frac{2\mu-4}{(1+\mu)^4}\right)\right\}.$$

Then, according to [1] or [8],

$$\Omega_- = -\frac{\exp(-\omega_- x)}{2\omega_-}\left\{2(\omega_+')^2 + s^2\left(\frac{4-2(1-\delta)}{(2-\delta)^4}\right)\right\} - x\exp(-\omega_- x)(\omega_+')^2 \tag{6.15a}$$

and

$$\Omega_+ = -\frac{\exp(-\omega_+ x)}{2\omega_+}\left\{s^2\left(\frac{4-2(1+\delta)}{(2+\delta)^4}\right)\right\}, \tag{6.15b}$$

where

$$\omega_- = \left[1+s+\frac{1-\delta}{(2+\delta)^2}s^2\right]^{\frac{1}{2}}, \quad \omega_+ = \left[1+s+\frac{1+\delta}{(2-\delta)^2}s^2\right]^{\frac{1}{2}},$$

$$\omega_+' = \frac{\delta}{(2-\delta)^3}\frac{s^2}{\omega_-} = -\omega_-',$$

since $(\partial\omega/\partial\mu)^2 \in [0, (\omega_+')^2]$.

The inverse Laplace transform can be simplified if $0 < \delta \ll 1$ by expanding the square root as

$$\left[1+s+\frac{1-\delta}{(2+\delta)^2}s^2\right]^{\frac{1}{2}} \approx \gamma\left[(s+2)+\frac{s+1}{2(s+2)}\left(\frac{(2+\delta)^2}{1-\delta}-4\right)\right]$$

with

$$\gamma := \left[\frac{1-\delta}{(2+\delta)^2}\right]^{\frac{1}{2}},$$

The calculation of this remark is typical of an interval evaluation of remainder terms.

6.3 *Sensitivity analysis for a finite line with zero boundary data*

The problem of (6.1) is studied again for the auxiliary conditions

$$e(x,0;\mu) = f(x), \quad e_t(x,0;\mu) = e(0,t;\mu) = e(1,t;\mu) \equiv 0,$$

on the domain $0 < x < 1, 0 < t < \infty$.

The classical separation of variables yields the solution

$$e(x,t;\mu) = \exp\left[-(1+\mu)^2 t/2\mu\right] \sum_{j=1}^{\infty} A_j \sin \omega_j x \cos \delta_j t, \tag{6.16}$$

where the A_j, $j = 1, 2, \ldots$ are the Fourier-sine coefficients for $f(x)$, which are independent of μ, $\omega_j^2 := j^2\pi^2$ are the eigenvalues and

$$\delta_j^2 := \frac{(\mu+1)^2}{\mu}\left[j^2\pi^2 - \frac{(\mu-1)^2}{4\mu}\right]. \tag{6.17}$$

We suppose μ is such that $\pi^2 > (\mu - 1)^2/(4\mu)$ to avoid the situation where linear or real exponential solutions arise during the separation process. (This holds for μ on the range $0 < \mu < (\approx 45)$).

Expansion of the exact solution for $e(x,t;\mu)$ in powers of $(\mu - 1)$ gives the solution

$$e(x,t,\mu) = e^{-2t}\left\{\sum_{j=1}^{\infty} A_j \sin j\pi x \cos 2j\pi t + \right.$$
$$+ \frac{(\mu-1)^2}{2}\, t \left[\sum_{j=1}^{\infty} A_j \sin j\pi x \cos 2j\pi t\right.$$
$$\left.\left. - \sum_{j=1}^{\infty} A_j \left(\frac{j^2\pi^2 - 1}{2j\pi}\right) \sin j\pi x \sin 2j\pi t\right]\right\}.$$

Remark 5. Again there is no first-order dependence on $(\mu - 1)$ indicating that the solution is insensitive to sufficiently small changes of the physical parameter $\mu = RC/(LG)$.

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